Application of Self-Affine Set Boundary Dimensions to Fractal Supply Chain Risks

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Abstract: Supply chains with fractal features can effectively reduce management risk, and the conclusions on self-similar fractal sets are generalized to self-affine sets on the basis of the method for evaluating the degree of risk of fractal supply chains. This paper introduced a method of calculating the boundary dimension of the self-affine set to study the degree of management risk of supply chains with fractal structure by comparing the boundary dimension of the self-affine set.

1. Introduction

Supply chain risk management plays an increasingly important role in global trade. As the global expansion of the supply chain increases its volatility, more internal and external risks emerge.

Ni Shenbing (2004) proposed the concept of fractal supply chain and argued that supply chain risk structure with fractal characteristics is an effective way to solve supply chain risk management [1]. The supply chain system is compressed into self-similar sets by utilizing the fractal characteristics.

Each node in a fractal supply chain is a fractal element. Its self-similar nature leads to self-organization, self-collaboration and self-optimization, which improves the efficiency and capability of the system and makes the fractal supply chain both stable and flexible. This is also the biggest characteristic of fractal supply chain that distinguishes it from previous supply chains.

Fractal dimension is a measure to characterize the complexity of objective things. Ni Shenbing (2004) characterized a fractal supply chain as a compressed self-similar set. The fractal dimension of the set delineates the degree of discrete variability in the N-dimensional space in which it is located, thus giving a comparison of the risk level of two supply chains.

The method is adapted to simple self-similar sets and cannot study more complex supply chains. Further research is needed when self-affine sets arise for fractal supply chains.

Ai gengyun (2009) generalized the method and introduced the generalized fractal dimension to analyze multiple fractals and their characteristic parameters. He also analyzed the effect of self-similarity on the revenue of each party under uncertain demand for two symmetric suppliers [2].

In this paper, we applied the conclusions of self-similar sets to self-affine sets, bypassing the complexity of the discussion by introducing an algorithm for the boundary dimensions of self-affine sets. The degree of risk in supply chain management with fractal structure is investigated by comparing the boundary dimensions of self-affine sets.

2. Preparatory Knowledge

2.1 The relationship between self-similarity and supply chain risk

A fractal supply chain consists of a number of different fractal elements that share various platform foundations of the organization. A higher degree of self-similarity among fractal elements means that the more resources they can share [3]. The higher the degree of self-similarity among fractal elements, the better the coordination in the mode of operation [4]. The degree of self-similarity between fractal elements in terms of management structure as well as organizational culture is also important for supply chain performance.

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2.2 Fractal supply chain risk evaluation

The degree of risk in a fractal supply chain can be delineated by multidimensional indicators. We set six dimensions as evaluative indicators: organizational structure, institutional procedures, calibration and reinforcement system, corporate culture, employee development, and information system. Their values are compared two by two by experts. Assuming that six experts evaluate an evaluative indicator, their evaluation results can be obtained as the following evaluation matrix:

c $m(x,y)=\begin{Bmatrix} m_{11} & m_{12} & \cdots & m_{16} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ m_{61} & m_{62} & \cdots & m_{66} \end{Bmatrix}_{6\times 6}$, where m_{ij} is the evaluation value given by the ith expert for the jth objective

In [1] the value of each item in the evaluation matrix, was processed as a set of points on the coordinate axes in the N-dimensional space to obtain the self-similarity set, the number of fractal dimensions was derived, and then the similarity deviation was derived to obtain the degree of similarity between fractal elements.

But the method has limitations when the supply chain structure is a complex self-similar set or self-affine set.

In most cases, the dimension of the self-affine set boundary reflects the complexity of the set as a whole. The more complex the self-mimicking set corresponding to the entity structure, the higher the risk of substitution of the supply chain. Therefore, we can take the method of calculating the boundary dimension of the self-affine set to determine the risk level of the two supply chains.

2.3 Basic concept

In the space E of compositions of nonempty compact subsets on R^d, $\{L_n\}_{n=1}^{N}$ is a system of function iterations, and with respect to the Hausdorff metric, we define a compression mapping L, $L(X)=\bigcup_{n=1}^{N} L_n(X), X$ on $E$to be an arbitrary nonempty compact set, then E has a unique attractor F.

$F=F(A,D)$ is the self-affine set, where A is the self-affine expansion matrix of all eigenvalues modulo greater than 1, and F_0 is the unit region centered at the origin and bounded parallel to the coordinate axes.

Then there exists L: $E\to E, L(F_0)=\bigcup_{i=1}^{N} L_i(F_0)=\bigcup_{i=1}^{N} A^{-1}(x+d_i), d_i\in D.$, so $\lim_{n\to\infty} F_n = F$.

Here, we used the method of correlation matrices to construct the evaluation matrix in 2.2.

For any $x\in N, d\in D$, such that x_d denotes the unique solution of $d+x\in Ax_d+D$; such that $k=|N|$ denotes the number of elements in N; and such that $m(x,y)$ denotes a matrix of order $n\times n$. Each term in $m(x,y)$ has the following form:

For $x,y\in N$, $m_{xy}=|\{d\in D|\, x+d=Ay+d_i, d_i\in D\}|$, constructing the first indicator of $m(x,y)$ corresponding to element $1\in N$, there are $m_{11}=|D|$ and when $y\neq 0$, $m_{1y}=0$. It makes the first column elements of $m(x,y)$ all 0 except for one item that is non-zero. $m(x,y)$ then denotes the $(n-1)\times(n-1)$-order matrix excluding the first row and the first column.

3. Main Theorems and Proof

Theorem 1 If the following equivalence condition is satisfied:

(1) $\lim_{m\to\infty}\partial F_m=\partial F$, (2) $\lim_{n\to\infty}\partial F_m$ has no interior point, (3) $m(F)=1$, (4) $\{F+x|\, x\in Z^d\}$ is a self-affine set of R^d [5]. Then there exists a constant Then there exists a constant A such that $d\,(\partial F,\partial F_n)\le ac^{-n}$:

Theorem 2 Let $\lim_{m\to\infty}\partial F_m = \partial F$, $F = F(A, D)$ is a self-affine set, and c is the expansion factor of A. The association matrix $m(x, y)$ has a maximal eigenvalue λ. Then, under the equivalence condition of Theorem 1, there is $\dim_E(\partial T) = \log\lambda / \log c$.

PROOF: F_0 is a unit region with center at the origin and edges parallel to the coordinate axes. $F_n = L^n(F_0)$ is an n-weight approximation of the self-affine set F. In the self-affine set F, the function iteration system $L: E \to E, L(F_0) = \bigcup_{i=1}^{N} L_i(F_0) = \bigcup_{i=1}^{N} A^{-1}(x + d_i), d_i \in D.$

Therefore, $F_n = \bigcup\{A^{-n}(F_0 + d_0 + Ad_1 + \cdots + A^{n-1}d_{n-1} \mid d_i \in D\}$, F_n is a parallel set after translation of the parts of $A^{-n}(T_0)$ and $A^{-i}(F_0) \cap A^{-j}(F_0) = \varphi, i \neq j$and $1 \le i, j \le N$. MapA^n establishes a bijection from a set consisting of small cubes in F_n into D [6].

For a given even pair (A, D), let $N' = N(A, D) \setminus \{0\}$, for any matrix M such that |M| denotes the sum of all entries in M. $|C^n|$ denotes the number of (x, y, d), and (x, y, d) satisfies $x + d = A^n y + d, d \in D$ [7]. $x, y \in N, d \in D$. B_n denotes the set consisting of elements d that satisfy $d + x \in A^n y + d, d \in D$ for some $x, y \in N$. β_n denotes the base of B_n, then we have $\beta_n \le |C^n| \le (k-1)^2\ \beta_n(2)$, and k is the base of N.

So *maps to* A^n is bijective and B_n corresponding to the set consisting of small cubes in F_n is also called B_n. Hence there is $N + D \subseteq AN + D$ which, by induction, gives $D + N \subseteq A^n N + D$.

$b = b(A, D)$ denotes the maximum Euclidean distance from the origin to any point in the neighborhood $N(A, D)$. By the foregoing proof and the conclusion of Theorem 1, the center of every cube in B_n is not more than $(a+b)/c^n$ from some point in ∂F, and similarly every point in ∂F is not more than $(a+b)/c^n$ from the center of such a cube. It is clear that the number of small cubes with side lengths of $1/c^n$ not more than a fixed value of distance from a fixed point in ∂F is finite, with respect to the parameter h. [8]

Let α_n be the minimum number of small cubes of side length $1/c^n$ covering ∂F.

$\beta_n \le \mathrm{h}\,\alpha_n$ and $\alpha_n \le \mathrm{h}\,\beta_n$. By equation (2), there are two positive constants a' and b', $a' = \frac{1}{h(k-1)^2}$, $b' = $ h, such that $a'|C^n| \le \alpha_n \le b'|C^n|$.

So $\lim_{n\to\infty}\frac{1}{n}\log(|C^n|) = \log\lambda$, and by [8], ∂F is a self-similar set, so the box dimension of ∂F agrees with the Hausdorff dimension pair ∂F [9], so $\underline{\dim}_B\partial F = \overline{\dim}_B\partial F = \dim_H\partial F$, and there is

$$\dim_H\partial F = \dim_B\partial F = \lim_{n\to\infty}\frac{\log(\alpha_n)}{-\log c^{-n}} = \lim_{n\to\infty}\frac{\log(\alpha_n)}{\log c^n} = \lim_{n\to\infty}\frac{\log|C^n|}{n\log c} = \frac{\log\lambda}{\log c}.$$

4. Application Examples

We applied the above methodology to analyze the riskiness of the supply chain being a self-affine set of pad maps and Twin dragon.

4.1 The gasket

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $D = \{(0,0),(1,0),(0,1),(-1,-1)\}$ we used the method showed above $N + D = AN + D$, Substituting the elements in D gives $N = \{(0,0),(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1)\}$, and calculating the evaluation matrix:

$$C_{11} = |d|,\ \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + d,\quad \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + d$$

$$\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+d,\ \begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}-1\\-1\end{pmatrix}=\begin{pmatrix}2&0\\0&2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}+d,\ d\in D$$

It can be obtained that only $d=(1,0)$,which makes the equation hold, so $C_{11}=1$, and the evaluation matrix can be obtained:

$$m(x,y)=\begin{pmatrix}1&1&0&0&0&1\\1&1&1&0&0&0\\0&1&1&1&0&0\\0&0&1&1&1&0\\0&0&0&1&1&1\\0&0&0&1&1&1\end{pmatrix}$$

It is easily obtained, the corresponding eigenvalue $\lambda=3$, $\dim_H \partial F=\lim_{n\to\infty}\frac{\log|C^n|}{n\log c}=\frac{\log\lambda}{\log c}=\frac{\log 3}{\log 2}\approx 1.58.$

4.2 Twin dragon

Let $A=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$, $D=\{(0,0),(1,0)\}$ we used the method showed above $N+D=AN+D$, Substituting the elements in D gives $N=\{(0,0),(0,1),(1,0),(1,-1),(0,-1),(-1,0),(-1,1)\}$, From the algorithm in 1, we can get $\dim_H \partial F\approx 1.52.$

Through the above comparison we can see that it is obvious that the fractal supply chain is close to the degree of risk for the gasket and for twin dragon, and the supply chain in twin dragon form is less risky.

5. Summary

This study introduces a new method to analyze supply chain risks by applying the concept of self-affine fractal sets. It innovates in calculating the boundary dimensions of these sets, offering a fresh perspective in understanding the complexity of fractal-characterized supply chains. This approach is significant in practical applications, aiding in the optimization of supply chain management. The paper also paves the way for future research, especially in applying these techniques to more complex supply chain structures, enhancing their real-world applicability.

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